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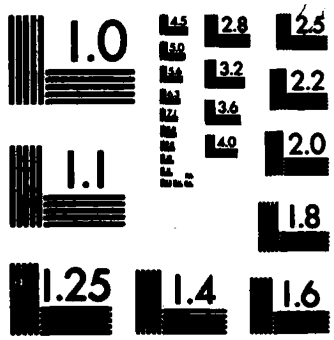

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A Measure of Variability Based on the Harmonic Mean, and its Use in Approximations

by Mark Brown
The City College, CUNY

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Let X be a positive variable and assume that both $a = EX^{-1}$ and $\mu = EX$ are finite. Define $c^2 = 1-(a\mu)^{-1}$. This quantity serves as a measure of variability for X which is reflected in the behavior of completely monotone functions of X. For g completely monotone with $g(0) < \infty$:

$$0 \leq Eg(X)-g(EX) \leq c^2 g(0) \qquad \text{Var } g(X) \leq c^2 g^2(0)$$

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Summary. Let X be a positive random variable and assume that both $a = EX^{-1}$ and $\mu = EX$ are finite. Define $c^2 = 1-(a\mu)^{-1}$. This quantity serves as a measure of variability for X which is reflected in the behavior of completely monotone functions of X. For g completely monotone with $g(0) < \infty$:

$$0 \leq Eg(X)-g(EX) \leq c^2 g(0)$$

$$\text{Var } g(X) \leq c^2 g^2(0)$$

1. Introduction. Given a random variable X and a function g, crude approximations to the mean and variance of $g(X)$ are obtainable by Taylor series arguments. The variance of X, σ^2, is a key quantity under this approach, both in approximating the bias $(Eg(X)-g(EX))$ and the variance $(\mathrm{Var}(g(X)))$. For X positive and g rapidly decreasing, the bias and variance of $g(X)$ should be relatively insensitive to the tail behavior of X, and σ^2 should therefore not play an important role. In practice, when σ^2 is very large the approximations for rapidly decreasing functions are often very poor.

We take the point of view that σ^2 is not measuring that aspect of variability which is relevant to the behavior of rapidly decreasing functions of X. For this purpose an often more informative measure of variability is:

$$c^2 = 1-(EXEX^{-1})^{-1} . \tag{1.1}$$

For X positive and g completely monotone we derive:

$$0 \le [Eg(X)-g(EX)]/g(0) \le c^2 \tag{1.2}$$

$$\mathrm{Var}[g(X)/g(0)] \le c^2 . \tag{1.3}$$

Thus if EX^{-1} is close to $(EX)^{-1}$ (as measured by c^2) then for g completely monotone, $g(X)/g(0)$ is close to a one point distribution at $g(EX)/g(0)$.

The quantity c^2 has an additional interpretation. For a positive random variable X with distribution F, consider a stationary renewal process on the whole real line, with interarrival time distribution F. Define T to be the length of the interval which covers $\{0\}$, and $V = T^{-1}$. Then:

$$\sigma_V^2 = \frac{EXEX^{-1}-1}{(EX)^2} = \frac{EX^{-1}}{EX} c^2 \tag{1.4}$$

$$\sigma_V^2/EV^2 = c^2 \ . \tag{1.5}$$

Moreover, defining $h(x) = (EX)xg(x^{-1})$, it follows that $g(EX) = h(EV)$ and:

$$Eg(X) = Eh(V) \ . \tag{1.6}$$

Use of (1.6) and a Taylor series argument yields:

$$0 \leq Eg(X)-g(EX) \leq \frac{c^2}{2} EX^{-1}\sup(x^3 g''(x)) \ . \tag{1.7}$$

Expressions (1.2) and (1.7) are competing inequalities. For X fixed with $EX^{-1} < \infty$, (1.2) will be better for some choices of g, and (1.7) for others.

2. <u>Definitions and preliminary results</u>. A function g on $[0,\infty)$ is defined to be completely monotone if it possesses derivitives of all orders and $(-1)^n g^{(n)}(\lambda) \geq 0$ for all $\lambda > 0$. In the above $g^{(0)} = g$. Some examples are $(x+a)^{-k}$ with $a \geq 0$, $k > 0$, $e^{-\alpha x}$ with $\alpha > 0$, and $e^{-\lambda(1-e^{-x})}$ with $\lambda > 0$. Lemma (2.1) below is due to Bernstein. An interesting discussion and proof of Bernstein's theorem is given in Feller [2] p. 439.

<u>Lemma 2.1</u>. A function g on $[0,\infty)$ is completely monotone with $g(0) = m < \infty$ if and only if it is of the form

$$g(x) = \int_0^\infty e^{-\lambda x} dH(x) \tag{2.2}$$

where H is a positive measure on $[0,\infty)$ with $H([0,\infty)) = m$.

Consider a probability distribution F on $[0,\infty)$. Its Laplace transform:

$$\mathcal{L}(\alpha) = \int_0^\infty e^{-\alpha x} dF(x) \tag{2.3}$$

is the survival function of a mixture of exponential distributions, i.e. if $Z|X=x$ is exponential with parameter x, then $\Pr(Z > \alpha | X=x) = e^{-\alpha x}$ and $\Pr(Z > \alpha) = \mathcal{L}(\alpha)$. With this observation, Lemma 2.2 below follows from Brown [1], theorem 4.1 part (xii).

Lemma 2.2. Let $\mathcal{L}$ be the Laplace transform of a probability distribution on $[0,\infty)$ with $a = \int_0^\infty \mathcal{L}(\alpha)d\alpha < \infty$ and $\mu = -\mathcal{L}'(0) < \infty$. Then:

$$0 \leq \mathcal{L}(\alpha)-e^{-\alpha\mu} \leq 1-(a\mu)^{-1} \text{ for all } \alpha \geq 0 . \tag{2.4}$$

3. Derivation of inequalities. Consider a positive random variable X with $a = EX^{-1}$ assumed finite, as well as $\mu = EX$. Define $c^2 = 1-(a\mu)^{-1}$. Note that $0 \leq c^2 < 1$ with equality if and only if X is a constant.

Theorem 3.1. Let g be a completely monotone function on $[0,\infty)$ with $g(0) < \infty$. Then:

$$0 \leq Eg(X)-g(\mu) \leq c^2 g(0) \tag{3.2}$$

$$\mathrm{Var}(g(X)) \leq c^2 g^2(0) . \tag{3.3}$$

Proof. By Lemma 2.2,

$$0 \leq \mathcal{L}(\alpha)-e^{-\alpha\mu} \leq c^2 \text{ for all } \alpha \geq 0 . \tag{3.4}$$

Since g is completely monotone, by Lemma 2.1 there exists a measure H on $[0,\infty)$ with $H[0,\infty) = g(0)$ and:

$$g(x) = \int e^{-\alpha x} dH(\alpha) . \tag{3.5}$$

Now:

$$(3.6) \qquad Eg(X) = \iint e^{-\alpha x} dH(\alpha) dF(x) = \int \mathcal{L}(\alpha) dH(\alpha)$$

$$(3.7) \qquad g(\mu) = \int e^{-\alpha\mu} dH(\alpha) \ .$$

Since g is convex, $Eg(X) \geq g(\mu)$. Thus from (3.4) and (3.6):

$$(3.8) \qquad 0 \leq Eg(X)-g(\mu) = \int_0^\infty (\mathcal{L}(\alpha)-e^{-\alpha\mu}) dH(\alpha) \leq c^2 g(0) \ .$$

Since g is completely monotone so is g^2 (Feller [2], p. 441). Applying (3.8) to g^2 we obtain:

$$(3.9) \qquad 0 \leq Eg^2(X)-g^2(\mu) \leq c^2 g^2(0) \ .$$

From (3.8) and (3.9), $\mathrm{Var}(g(X)) = Eg^2(X)-(Eg(X))^2 \leq (g^2(\mu)+c^2g^2(0))-g^2(\mu) = c^2g^2(0)$. This concludes the proof.

Given $X > 0$ with distribution F, consider a stationary renewal process on the whole real line with interarrival time distribution F. Define T to be the length of the interval containing 0. It follows from Feller [2] p. 371 that:

$$(3.10) \qquad dF_T(x) = x\, dF(x)/\mu \ .$$

From (3.10) we see that $ET^{-1} = \mu^{-1}$ and $ET^{-2} = a\mu^{-1}$ where $a = EX^{-1}$. Defining $V = T^{-1}$ it follows that $\sigma_V^2 = (a\mu-1)\mu^{-2} = c^2 a\mu^{-1}$, while

$\sigma_V^2/EV^2 = c^2$. Also from (3.10) we see that $Eg(X) = E(\mu V g(V^{-1})) = Eh(V)$ where $h(x) = \mu x g(x^{-1})$. Note that $h(EV) = h(\mu^{-1}) = g(EX)$. Finally since:

$$(3.11) \qquad h(V) = h(EV) + (V-EV)h'(EV) + \frac{(V-EV)^2}{2} h''(V^*)$$

with V^* between EV and V, it follows that:

$$(3.12) \quad Eg(X) \leq h(EV) + \frac{\sigma_V^2}{2} \sup(h''(x)) = g(\mu) + \frac{c^2 a \mu^{-1}}{2} \sup(h''(x)) \ .$$

But $h''(x) = \mu x^{-3} g''(x^{-1})$, and thus $\sup h''(x) = \mu \sup(x^3 g''(x))$. Thus from (3.12)

$$(3.13) \qquad 0 \leq Eg(X) - g(\mu) \leq \frac{c^2}{2} a \sup(x^3 g''(x)) \ .$$

References

[1] Brown, M. (1981). "Approximating IMRL distributions by exponential distributions, with applications to first passage times." To appear in Ann. Probability.

[2] Feller, W. (1971). An Introduction to Probability Theory and its Applications, Volume II, 2nd Edition, John Wiley, New York.